Note Title 01-11-2016

Particle on a ring

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi+V\psi=E\psi$$

For particle on a ring, $\theta=\pi/2$, r is fixed, $V=0$

$$-\frac{\hbar^2}{2m}\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi=E\psi$$

or $$-\frac{\hbar^2}{2mr^2}\frac{\partial^2\psi}{\partial\phi^2}=E\psi$$

$$\frac{\partial^2\psi}{\partial\phi^2}=-\underbrace{\frac{2IE}{\hbar^2}}\psi$$

$I=mr^2$ (moment of inertia)

$$m_l^2=\frac{2IE}{\hbar^2}$$

$$\Phi(\phi)=a_{m_l}e^{im_l\phi}\quad\ldots\quad (1)$$

$$E_{m_l}=\frac{m_l^2\hbar^2}{2I}$$

$$\Phi(\phi)=\Phi(\phi+2\pi)$$

$$m_l=0,\pm1,\pm2,\pm3\ \ldots$$

quantum number

$$\int_0^{2\pi} \Phi^*(\phi)\,\Phi(\phi)\,d\phi = 1$$

$$a_m = \frac{1}{\sqrt{2\pi}}$$

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{i m_l \phi} \quad \cdots \quad (2)$$

$$\cancel{\Phi(\phi)} \quad \Phi^*(\phi)\,\Phi(\phi) = \frac{1}{2\pi}$$

Angular momentum

$$L = r \times p = \begin{vmatrix} i & j & k \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix}$$

$$i\underbrace{(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)}_{\hat{l}_x} - j\overbrace{(\hat{x}\hat{p}_z - \hat{z}\hat{p}_x)}^{-\hat{l}_y}$$
$$+ k\underbrace{(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)}_{\hat{l}_z}$$

$$\hat{l}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$= \hat{x}\left(-i\hbar\frac{\partial}{\partial y}\right) - \hat{y}\left(-i\hbar\frac{\partial}{\partial x}\right)$$

$$= -i\hbar\left(\hat{x}\frac{\partial}{\partial y} - \hat{y}\frac{\partial}{\partial x}\right)$$

$$\hat{l}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{l}_z \, \Phi(\phi) = \hat{l}_z \, \cancel{\phi} \, \frac{1}{\sqrt{2\pi}} e^{i m_l \phi}$$

$$= -i\hbar \frac{\partial}{\partial \phi} \frac{1}{\sqrt{2\pi}} e^{i m_l \phi}$$

$$= \underbrace{m_l \hbar}_{\text{eigen value}} \, \Phi(\phi)$$

Particle on a sphere:

no change in 'r' $\Rightarrow$ Rigid rotor

$V = 0$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi$$

$$= E\,\psi(\theta,\phi)$$

$$-\frac{\hbar^2}{2mr^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi$$

$$= E\,\psi(\theta,\phi) \quad \cdots \quad (1)$$

$$\psi(\theta,\phi) \Rightarrow Y(\theta,\phi)$$

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y=-\frac{2IE}{\hbar^2}Y(\theta,\phi)$$

$$\ldots \quad (2)$$

$$\beta=\frac{2IE}{\hbar^2}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta\Phi+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Theta\Phi=-\beta\underbrace{Y(\theta,\phi)}_{\Theta\Phi}$$

$$\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta}+\frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\beta\Theta\Phi$$

Divide throughout by $\Theta\Phi$

Multiply throughout by $\sin^2\theta$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta}+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-\beta\sin^2\theta$$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta}+\beta\sin^2\theta=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \quad (2)$$